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Bending and stability analysis of gradient elastic beams

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Abstract

The problems of bending and stability of Bernoulli–Euler beams are solved analytically on the basis of a simple linear theory of gradient elasticity with surface energy. The governing equations of equilibrium are obtained by both a combination of the basic equations and a variational statement. The additional boundary conditions are obtained by both variational and weighted residual approaches. Two boundary value problems (one for bending and one for stability) are solved and the gradient elasticity effect on the beam bending response and its critical (buckling) load is assessed for both cases. It is found that beam deflections decrease and buckling load increases for increasing values of the gradient coefficient, while the surface energy effect is small and insignificant for bending and buckling, respectively.

Keywords: Microstructural effects; Gradient elasticity; Surface energy; Bending of beams; Stability of beams; Variational principle; Non-classical boundary conditions

1. Introduction

The mechanical behavior of linear elastic materials with microstructure, such as polymers, polycrystals or granular materials, cannot be described adequately by the classical theory of linear elasticity, which is associated with the concepts of homogeneity and locality of stress. When the material exhibits a non-homogeneous behavior, microstructural effects are important and the state of stress has to be defined in a non-local manner. These microstructural effects can be successfully modeled in a macroscopic manner by employing higher-order gradient, micropolar and couple stress theories. For a literature review on the subject of these theories one can consult the review articles of Tiersten and Bleustein (1974) and Exadaktylos and Vardoulakis (2001), the book of Vardoulakis and Sulem (1995) and the literature review in the recent paper by Tsepoura et al. (2002).

These theories, usually in simplified forms, have been used during the last fifteen years or so to successfully solve various boundary value problems of static and dynamic linear elasticity. Thus, it has been

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found that singularities or discontinuities of classical elasticity theory disappear, size effects are easily captured and wave dispersion effects are observed in cases where this was not possible in classical elasticity (e.g. Tiersten and Bleustein, 1974; Exadaktylos and Vardoulakis, 2001; Vardoulakis and Sulem, 1995; Tsepoura et al., 2002; Altan and Aifantis, 1992; Ru and Aifantis, 1993; Altan et al., 1996; Exadaktylos et al., 1996; Chang and Gao, 1997; Georgiadis and Vardoulakis, 1998).

In this paper the problems of bending and buckling of Bernoulli–Euler beams are solved analytically on the basis of a simple theory of gradient elasticity with surface energy. The governing equations of equilibrium for both bending and buckling problems are derived both by combining the corresponding basic equations and by using a variational statement. All possible boundary conditions (classical and non-classical) are obtained with the aid of a variational statement constructed by both the establishment of an expression for the strain energy and the use of the method of weighted residuals. In addition, boundary value problems of bending and buckling of beams are solved analytically and the gradient effect on the response of the beam or its critical (buckling) load is assessed.

The problem of bending of beams has been studied by non-classical theories of elasticity mainly in order to explain test results, which could not be explained by classical elasticity theory. Thus, Krishna Reddy and Venkatasubramanian (1978) determined analytically the flexural rigidity of circular cylindrical beams of Cosserat (micropolar elastic) material, while Lakes (1983, 1986, 1995) and Anderson and Lakes (1994) investigated the dependence of the flexural rigidity of rods, made of various polymeric foams, upon specimen size both experimentally and by using the Cosserat (micropolar elasticity) theory. Vardoulakis et al. (1998) studied the effect of the beam length on the failure load and the variation of the beam curvature along the beam length both experimentally and on the basis of a gradient theory with surface energy for Timoshenko beams in flexure. Tsagrakis (2001) briefly considered the case of pure bending of elastic Bernoulli–Euler rods and verified the test results of Lakes (1983, 1986) by using the simple gradient elasticity theory of Aifantis and coworkers (1992, 1993) and a gradient elasticity theory with surface energy. Thus, the present paper presents a more systematic and general treatment of bending of beams than in Lakes (1983, 1986, 1995), Anderson and Lakes (1994), Vardoulakis et al. (1998), Tsagrakis (2001) and in addition considers buckling of beams.

2. Governing equation and boundary conditions for bending by basic equations and a variational principle

Consider a straight prismatic beam, which is subjected to a static lateral load $q(x)$ distributed along the longitudinal axis x of the beam, as shown in Fig. 1. Thus the loading plane coincides with the yx plane. The cross-section of the beam A is characterized by the two axes y and z with the former one being its axis of symmetry.

In this work the simple gradient elasticity theory with surface energy due to Vardoulakis and Sulem (1995) is employed. This theory combines the general concepts, ideas and structure of Mindlin's (1964) theory with Casal's (1972) concept on surface energy effects and is associated with only four elastic con-

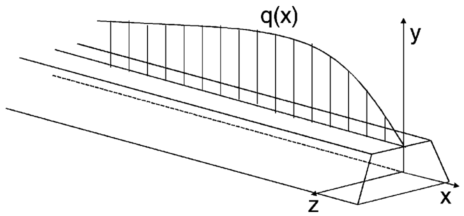

Fig. 1. Geometry and loading of a prismatic beam in bending.

stants (two classical and two non-classical) instead of the 18 elastic constants (including the two classical ones) of Mindlin's (1964) theory. Thus, the present theory of Vardoulakis and Sulem (1995), because of its simplicity is much more convenient in applications than Mindlin's (1964) gradient elasticity theory with 18 elastic constants and the Cosserat and Cosserat (1909) or the micropolar elasticity theory of Eringen (1966) with six elastic constants. Thus, on the basis of the simple theory of gradient elasticity with surface energy (Vardoulakis and Sulem, 1995), one has for the one-dimensional case that the Cauchy and double stresses as well as the total stresses τ_x, μ_x and σ_x, respectively, are given for the case of beam bending by the constitutive relations

$$\tau_x = Ee_x + \ell Ee'_x \tag{1}$$

$$\mu_x = \ell Ee_x + g^2 Ee'_x \tag{2}$$

$$\sigma_x = \tau_x - \frac{\mathrm{d}\mu_x}{\mathrm{d}x} = E\left(e_x - g^2\frac{\mathrm{d}^2 e_x}{\mathrm{d}x^2}\right) = E(e_x - g^2 e''_x) \tag{3}$$

where e_x represents the axial strain of the beam in bending, the constants ℓ and g^2 represent material lengths related to surface and volumetric elastic strain energy, respectively, E is the Young's modulus and primes indicate differentiation with respect to x. Since the strain energy is positive definite, the material lengths g^2 and ℓ are restricted, such that (Vardoulakis and Sulem, 1995), $0 < \ell < g^2$.

Conditions of equilibrium require that the resultant of the internal forces on the cross-section should be zero, and their moment equal to the bending moment M. Thus

$$\int_A \sigma_x \,\mathrm{d}A = 0 \tag{4}$$

$$\int_A \sigma_x y \,\mathrm{d}A = -M \tag{5}$$

with

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V, \quad \frac{\mathrm{d}V}{\mathrm{d}x} = -q(x) \tag{6}$$

where V represents shear forces.

In view of Eq. (3) and according to Bernoulli–Euler hypothesis (Timoshenko and Goodier, 1970) that $e_x = ky$, with k denoting the curvature along the x-direction, Eqs. (4) and (5) take the form

$$E\left(k - g^2\frac{\mathrm{d}^2 k}{\mathrm{d}x^2}\right)\int_A y\,\mathrm{d}A = 0 \tag{7}$$

$$E\left(k - g^2\frac{\mathrm{d}^2 k}{\mathrm{d}x^2}\right)\int_A y^2\,\mathrm{d}A = -M \tag{8}$$

Eqs. (7) and (8) are both satisfied for $\int_A y\,\mathrm{d}A = 0$, indicating that the x-axis is a centroidal one, and

$$k - g^2\frac{\mathrm{d}^2 k}{\mathrm{d}x^2} = -\frac{M}{EI} \tag{9}$$

where $I = \int_A y^2\,\mathrm{d}A$ stands for the moment of inertia about the z-axis of the beam.

Utilizing the Bernoulli–Euler's assumption (Timoshenko and Goodier, 1970) that

$$k = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \tag{10}$$

and Eqs. (6) and (10), Eq. (9) results in the governing equation of beam in bending

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}}) = -q(x)$$

or

$$EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}}) + q(x) = 0 \tag{11}$$

In this section, the governing equation of equilibrium of a gradient elastic beam in bending as well as the corresponding boundary conditions are also determined by means of a variational principle. Consider again the straight prismatic Bernoulli–Euler beam of Fig. 1. On the basis of the aforementioned Bernoulli–Euler assumptions, the equation of equilibrium of the gradient elastic beam in bending as well as all possible boundary conditions can be determined with the aid of the variational principle

$$\delta(U - W) = 0 \tag{12}$$

where U is the strain energy, W is the total work done by external forces and δ indicates variation. According to the one-dimensional gradient elasticity theory with surface energy (Vardoulakis and Sulem, 1995), the strain energy of a beam in bending is defined as

$$U = \frac{1}{2}\int_A \int_0^L [\tau_x \cdot e_x + \mu_x \cdot \nabla e_x]\,\mathrm{d}x\,\mathrm{d}A \tag{13}$$

where $e_x = -yu''$ represents the axial strain of the beam, $\nabla e_x = \mathrm{d}e_x/\mathrm{d}x = -yu'''$ stands for the strain gradient and τ_x and μ_x denote the Cauchy and double stresses given by Eqs. (1) and (2), respectively.

Substituting Eqs. (1) and (2) into Eq. (13) one obtains the following expression for the strain energy

$$U = \frac{1}{2}\int_0^L EI[(u'')^2 + g^2(u''')^2 + 2\ell u'' u''']\,\mathrm{d}x \tag{14}$$

According to the calculus of variations, the variation of an integral of the type $U = \int_0^L F(u'', u''')\,\mathrm{d}x$, is obtained through the well-known relation (Lanczos, 1970)

$$\delta U = \int_0^L \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial u''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u\,\mathrm{d}x + \left[\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u''}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u\right]_0^L + \left[\left[\frac{\partial F}{\partial u''} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u'\right]_0^L + \left[\frac{\partial F}{\partial u'''}\,\delta u''\right]_0^L \tag{15}$$

where, for the present case, the Lagrangian function F is

$$F = \frac{EI}{2}\left[(u'')^2 + g^2(u''')^2 + 2\ell u'' u'''\right] \tag{16}$$

Eqs. (15) and (16) help to express the variation of the strain energy of the beam as

$$\delta U = \int_0^L EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}})\,\delta u\,\mathrm{d}x + [EI(g^2 u^{\mathrm{V}} - u''')\,\delta u]_0^L + [EI(u'' - g^2 u^{\mathrm{IV}})\,\delta u']_0^L + [EI(\ell u'' + g^2 u''')\,\delta u'']_0^L \tag{17}$$

On the other hand, the variation of the work done by the external force $q(x)$, the boundary shear force V and the boundary classical and non-classical (double) bending moments M and m, respectively, reads

$$\delta W = -\int_0^L q\,\delta u\,\mathrm{d}x - [V\,\delta u]_0^L + [M\,\delta u']_0^L + [m\,\delta u'']_0^L \tag{18}$$

In view of Eqs. (17) and (18), the variational equation (12) takes the form

$$\begin{aligned}\delta(U-W) = \int_0^L [EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}}) + q]\,\delta u\,\mathrm{d}x + [\{V - EI(u''' - g^2 u^{\mathrm{V}})\}\,\delta u]_0^L \\ - [\{M - EI(u'' - g^2 u^{\mathrm{IV}})\}\,\delta u']_0^L - [\{m - EI(\ell u'' + g^2 u''')\}\,\delta u'']_0^L = 0\end{aligned} \tag{19}$$

The above variational equation implies that each term of Eq. (19) must be equal to zero. Thus, the governing equation of the beam in bending is given by

$$EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}}) + q(x) = 0 \tag{20}$$

which is the same as Eq. (11) derived with the aid of the basic equations, while the boundary conditions satisfy the equations

$$\begin{aligned}&[V(L) - EI[u'''(L) - g^2 u^{\mathrm{V}}(L)]]\,\delta u(L) - [V(0) - EI[u'''(0) - g^2 u^{\mathrm{V}}(0)]]\,\delta u(0) = 0\\ &[M(L) - EI[u''(L) - g^2 u^{\mathrm{IV}}(L)]]\,\delta u'(L) - [M(0) - EI[u''(0) - g^2 u^{\mathrm{IV}}(0)]]\,\delta u'(0) = 0\\ &[m(L) - EI[\ell u''(L) + g^2 u'''(L)]]\,\delta u''(L) - [m(0) - EI[\ell u''(0) + g^2 u'''(0)]]\,\delta u''(0) = 0\end{aligned} \tag{21}$$

For example, if one assumes the four classical boundary conditions to be $u(0)$, $u(L)$, $u'(0)$ and $u'(L)$ prescribed and the corresponding non-classical ones to be $u''(0)$ and $u''(L)$ prescribed, then $\delta u(0) = \delta u(L) = 0$, $\delta u'(0) = \delta u'(L) = 0$, $\delta u''(0) = \delta u''(L) = 0$ and Eqs. (21) are all satisfied. In view of Eqs. (21) one can observe that, when dealing with the classical boundary conditions, either the deflection u or the shear forces $V \equiv EI(u''' - g^2 u^{\mathrm{V}})$ and the strain u' or the bending moments $M \equiv EI(u'' - g^2 u^{\mathrm{IV}})$ at the boundary of the beam have to be specified. For the case of the non-classical or additional boundary conditions, one has to specify either the boundary strain gradient u'' or the boundary double moments $m \equiv EI(\ell u'' + g^2 u''')$.

3. Boundary conditions for bending through weighted residuals

In case where the equation of equilibrium or the equation of motion of a boundary value problem is known, then all possible boundary conditions of the problem can be also determined with the aid of the method of weighted residuals. This approach is particularly convenient in cases where an expression for the strain energy is not known or difficult to obtain. In this section, the boundary conditions corresponding to bending of a gradient elastic beam are determined by means of weighted residuals.

The lateral beam deflection $u(x)$ obeys the governing Eq. (11). Thus, according to the method of weighted residuals, one has the weak statement

$$\int_0^L (EIu^{\mathrm{IV}} - g^2 EIu^{\mathrm{VI}} + q)w\,\mathrm{d}x = 0 \tag{22}$$

where $w = w(x)$ is a weighting function. Integrating Eq. (22) by parts two and three times for u^{IV} and u^{VI}, respectively, one has

$$\int_0^L u^{\mathrm{IV}} w\,\mathrm{d}x = [u''' w - u'' w']_0^L + \int_0^L u'' w''\,\mathrm{d}x$$
$$\int_0^L u^{\mathrm{VI}} w\,\mathrm{d}x = [u^{\mathrm{V}} w - u^{\mathrm{IV}} w' + u''' w'']_0^L - \int_0^L u''' w'''\,\mathrm{d}x \tag{23}$$

Assuming that $w = \delta u$, where δ indicates variation, one obtains from Eqs. (22) and (23)

$$\int_0^L (EIu^{\mathrm{IV}} - g^2 EIu^{\mathrm{VI}} + q)\,\delta u\,\mathrm{d}x = EI[u'''\,\delta u - u''\,\delta u']_0^L - g^2 EI[u^{\mathrm{V}}\,\delta u - u^{\mathrm{IV}}\,\delta u' + u'''\,\delta u'']_0^L + EI\int_0^L u''\,\delta u''\,\mathrm{d}x + g^2 EI\int_0^L u'''\,\delta u'''\,\mathrm{d}x - \int_0^L q\,\delta u\,\mathrm{d}x$$

or

$$\frac{EI}{2}\delta\int_0^L [(u'')^2 + g^2(u''')^2 + qu]\,\mathrm{d}x = \int_0^L (EIu^{\mathrm{IV}} - g^2 EIu^{\mathrm{VI}} + q)\,\delta u\,\mathrm{d}x + EI[(u'' - g^2 u^{\mathrm{IV}})\,\delta u']_0^L - EI[(u''' - g^2 u^{\mathrm{V}})\,\delta u]_0^L + EI[g^2 u'''\,\delta u'']_0^L \tag{24}$$

Taking into account that the strain energy stored by the gradient elastic beam is that of Eq. (14), Eq. (24) leads to the expression

$$\frac{EI}{2}\delta\int_0^L [(u'')^2 + g^2(u''')^2 + 2\ell u'' u''' + qu]\,\mathrm{d}x = \int_0^L (EIu^{\mathrm{IV}} - g^2 EIu^{\mathrm{VI}} + q)\,\delta u\,\mathrm{d}x + EI[(u'' - g^2 u^{\mathrm{IV}})\,\delta u']_0^L - EI[(u''' - g^2 u^{\mathrm{V}})\,\delta u]_0^L + EI[(\ell u'' + g^2 u''')\,\delta u'']_0^L$$

or

$$\delta U = \int_0^L (EIu^{\mathrm{IV}} - g^2 EIu^{\mathrm{VI}} + q)\,\delta u\,\mathrm{d}x + EI[(u'' - g^2 u^{\mathrm{IV}})\,\delta u']_0^L - EI[(u''' - g^2 u^{\mathrm{V}})\,\delta u]_0^L + EI[(\ell u'' + g^2 u''')\,\delta u'']_0^L \tag{25}$$

In view of Eqs. (22) and (25), the following variational statement is implied:

$$\delta U = -EI[(u''' - g^2 u^{\mathrm{V}})\,\delta u]_0^L + EI[(u'' - g^2 u^{\mathrm{IV}})\,\delta u']_0^L + EI[(\ell u'' + g^2 u''')\,\delta u'']_0^L \tag{26}$$

However, according to Eqs. (12) and (18), the presence of the boundary shear as well as the boundary classical and non-classical (double) moments transform the above relation to the equivalent one

$$[\{V - EI(u''' - g^2 u^{\mathrm{V}})\}\,\delta u]_0^L - [\{M - EI(u'' - g^2 u^{\mathrm{IV}})\}\,\delta u']_0^L - [\{m - EI(\ell u'' + g^2 u''')\}\,\delta u'']_0^L = 0 \tag{27}$$

Thus it is apparent from Eq. (27) that the boundary conditions satisfy the equations

$$[V(L) - EI[u'''(L) - g^2 u^{\mathrm{V}}(L)]]\,\delta u(L) - [V(0) - EI[u'''(0) - g^2 u^{\mathrm{V}}(0)]]\,\delta u(0) = 0$$
$$[M(L) - EI[u''(L) - g^2 u^{\mathrm{IV}}(L)]]\,\delta u'(L) - [M(0) - EI[u''(0) - g^2 u^{\mathrm{IV}}(0)]]\,\delta u'(0) = 0 \tag{28}$$
$$[m(L) - EI[\ell u''(L) + g^2 u'''(L)]]\,\delta u''(L) - [m(0) - EI[\ell u''(0) + g^2 u'''(0)]]\,\delta u''(0) = 0$$

which, as it is expected, are identical to Eqs. (21).

4. Governing equations and boundary conditions for buckling by a variational principle

In this section, the governing equation of equilibrium of a beam in buckling as well as the corresponding boundary conditions are determined by means of a variational principle. Consider the beam of the previous section without lateral load subjected to an axial compressive force P, which can cause flexural buckling for a certain value of P called elastic buckling load or critical load P_{cr} to be determined. The governing equation of a beam in buckling as well as all possible boundary conditions can be determined with the aid of a variational principle, which reads as in Eq. (12). The strain energy of the gradient elastic beam in bending is defined by Eq. (14). Considering in addition the effect of the axial compressive force P, one obtains the following expression for the strain energy:

$$U = \frac{1}{2}\int_0^L EI[(u'')^2 + g^2(u''')^2 + 2\ell u''u''']\,dx - \frac{1}{2}\int_0^L P(u')^2\,dx \tag{29}$$

According to the calculus of variations, the variation of an integral of the type $U = \int_0^L F(u', u'', u''')\,dx$ is obtained through the well-known relation (Lanczos, 1970)

$$\delta U = \int_0^L \left[-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u\,dx + \left[\left[\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u\right]_0^L + \left[\left[\frac{\partial F}{\partial u''} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'''}\right)\right]\delta u'\right]_0^L + \left[\frac{\partial F}{\partial u'''}\,\delta u''\right]_0^L \tag{30}$$

where, for the present case, the Lagrangian function F is

$$F = \frac{EI}{2}[(u'')^2 + g^2(u''')^2 + 2\ell u''u'''] - \frac{P}{2}(u')^2 \tag{31}$$

Eqs. (30) and (31) help to express the variation of the strain energy of the beam as

$$\delta U = \int_0^L [EI(u^{IV} - g^2u^{VI}) + Pu'']\,\delta u\,dx - [[Pu' + EI(u''' - g^2u^{V})]\,\delta u]_0^L + [EI(u'' - g^2u^{IV})\,\delta u']_0^L + [EI(\ell u'' + g^2u''')\,\delta u'']_0^L \tag{32}$$

On the other hand, the variation of the work done by the external force P, the boundary shear force V as well as the boundary classical and non-classical (double) moments M and m, respectively, reads

$$\delta W = -\int_0^L P\,\delta u\,dx - [V\,\delta u]_0^L + [M\,\delta u']_0^L + [m\,\delta u'']_0^L \tag{33}$$

In view of Eqs. (32) and (33), the variational equation (12) takes the form

$$\delta(U - W) = \int_0^L [EI(u^{IV} - g^2u^{VI}) + Pu'']\,\delta u\,dx + [\{V - [Pu' + EI(u''' - g^2u^{V})]\}\,\delta u]_0^L - [\{M - EI(u'' - g^2u^{IV})\}\,\delta u']_0^L - [\{m - EI(\ell u'' + g^2u''')\}\,\delta u'']_0^L = 0 \tag{34}$$

The above variational equation implies that each term of Eq. (34) must be equal to zero. Thus, the governing equation of the beam in buckling is given by

$$EI(u^{IV} - g^2u^{VI}) + Pu'' = 0 \tag{35}$$

while the boundary conditions satisfy the equations

$$\begin{aligned}&[V(L)-[Pu'(L)+EI[u'''(L)-g^2u^{\mathrm{V}}(L)]]]\,\delta u(L)-[V(0)-[Pu'(0)+EI[u'''(0)-g^2u^{\mathrm{V}}(0)]]]\,\delta u(0)=0\\&[M(L)-EI[u''(L)-g^2u^{\mathrm{IV}}(L)]]\,\delta u'(L)-[M(0)-EI[u''(0)-g^2u^{\mathrm{IV}}(0)]]\,\delta u'(0)=0\\&[m(L)-EI[\ell u''(L)+g^2u'''(L)]]\,\delta u''(L)-[m(0)-EI[\ell u''(0)+g^2u'''(0)]]\,\delta u''(0)=0\end{aligned}\tag{36}$$

5. Governing equations and boundary conditions for buckling by basic equations and weighted residuals

The governing equation for a beam in buckling (Eq. (35)) can be easily obtained with the aid of the basic equations of the problem. Thus one has simply to augment the bending Eq. (20) with $q=0$ with the effect of the axial compressive force P reading Pu''.

In case where the governing equation of the beam in buckling is known, all possible boundary conditions can also be determined with the aid of the method of weighted residuals.

According to the method of weighted residuals, one has the weak statement

$$\int_0^L (EIu^{\mathrm{IV}}-g^2EIu^{\mathrm{VI}}+Pu'')w\,\mathrm{d}x=0\tag{37}$$

The first two terms of the integrand of Eq. (37) are treated in the same way as in Section 3 (Eq. (22)). The third term is integrated by parts once. On the assumption that $w=\delta u$ and taking into account Eqs. (12), (18) and (37), the following variational statement is implied:

$$[\{V-[Pu'+EI(u'''-g^2u^{\mathrm{V}})]\}\,\delta u]_0^L-[\{M-EI(u''-g^2u^{\mathrm{IV}})\}\,\delta u']_0^L-[\{m-EI(\ell u''+g^2u''')\}\,\delta u'']_0^L=0\tag{38}$$

Eq. (38) leads to the boundary conditions that, as in the case of the variational principle approach, satisfy Eq. (36).

6. Solution of boundary value problems in bending

This section deals with the solution of a boundary value problem for bending. Consider a cantilever beam of length L with its built-in end at $x=0$, subjected to a static uniformly distributed lateral load q. As it is shown in Section 2, the deflection $u(x)$ of the beam in bending satisfies Eq. (20) or

$$EI(u^{\mathrm{IV}}-g^2u^{\mathrm{VI}})=-q(x)\tag{39}$$

The solution of Eq. (39) is the sum of the solution of its homogeneous part, i.e., the one with $q=0$, and a particular solution of Eq. (39). The former part of the solution is

$$u_{\mathrm{h}}=c_1x^3+c_2x^2+c_3x+c_4+c_5g^4\sinh(x/g)+c_6g^4\cosh(x/g)\tag{40}$$

The classical boundary conditions are $u(0)=u'(0)=0$ and $M(L)=V(L)=0$ implying, the first two that $\delta u(0)=\delta u'(0)=0$, and the second two that $u''(L)-g^2u^{\mathrm{IV}}(L)=u'''(L)-g^2u^{\mathrm{V}}(L)=0$. Thus, Eqs. $(23)_{1,2}$ are satisfied. The non-classical boundary conditions are assumed to be $u''(0)=u'''(L)=0$ in case of $\ell=0$ and $u''(0)=[\ell u''(L)+g^2u'''(L)]=0$ in case of taking into account the surface energy terms ($\ell\neq 0$). Use of the above boundary conditions in conjunction with a particular solution $u_{\mathrm{p}}=-(q/24EI)x^4$, enable one to determine the constants c_1–c_6 of the homogeneous part of the solution u_{h} given by Eq. (40). They are

$$c_1 = qL/6EI, \quad c_2 = -\frac{qL^2}{4EI}(2(c\cdot d)^2+1), \quad c_3 = \left(\frac{qc\cdot d}{2EI}\right)L^3(2(c\cdot d)^2+1)\tanh\left(\frac{1}{c\cdot d}\right),$$
$$c_4 = -\left(\frac{q(c\cdot d)^2}{2EI}\right)L^4(2(c\cdot d)^2+1), \quad c_5 = -\left(\frac{q}{4EI}\right)\left(2+\frac{1}{(c\cdot d)^2}\right)\left[-1+\tanh\left(\frac{L}{g}\right)\right], \tag{41}$$
$$c_6 = \left(\frac{q}{4EI}\right)\left(2+\frac{1}{(c\cdot d)^2}\right)\left[1+\tanh\left(\frac{L}{g}\right)\right]$$

for the case of $\lambda = L/g = 0$, and

$$c_1 = qL/6EI, \quad c_2 = -\frac{qL^2}{4EI}(2(c\cdot d)^2+1),$$
$$c_3 = \left(\frac{qL^3}{2EI}\right)(c\cdot d)\left[\frac{-2(c\cdot d)^2\lambda+\lambda(2(c\cdot d)^2+1)\cosh\left(\frac{1}{c\cdot d}\right)+(2(c\cdot d)^2+1)\sinh\left(\frac{1}{c\cdot d}\right)}{\cosh\left(\frac{1}{c\cdot d}\right)+\lambda\sinh\left(\frac{1}{c\cdot d}\right)}\right],$$
$$c_4 = -\left(\frac{q(c\cdot d)^2L^4}{2EI}\right)(2(c\cdot d)^2+1), \tag{42}$$
$$c_5 = \left(\frac{q}{2EI}\right)\left[\frac{1-\lambda+2(c\cdot d)^2\left(1+\left(-1+\mathrm{e}^{1/(c\cdot d)}\right)\lambda\right)}{(c\cdot d)^2(1-\lambda+\mathrm{e}^{2/(c\cdot d)}(1+\lambda))}\right],$$
$$c_6 = \left(\frac{q}{2EI}\right)\left[\frac{\mathrm{e}^{1/(c\cdot d)}\left(-2(c\cdot d)^2\lambda+\mathrm{e}^{1/(c\cdot d)}(1+\lambda)(2(c\cdot d)^2+1)\right)}{(c\cdot d)^2(1-\lambda+\mathrm{e}^{1/(c\cdot d)}(1+\lambda))}\right]$$

for the case of $\lambda = L/g \neq 0$.

Fig. 2(a) shows the variation of the beam deflection $\bar{u}(\xi)$ along the dimensionless distance $\xi = x/L$ for various values of the gradient coefficient product $c\cdot d = (g/D)\cdot(D/L)$, where D is a characteristic diameter of the microstructure. The value $c\cdot d = 0$ corresponds to the classical elastic case. Fig. 2(a) shows that the deflection of the gradient beam without surface energy decreases as the product $c\cdot d$ increases. Figs. 2(b) and (c) show the variation of the beam deflection $\bar{u}(\xi)$ along the ξ for various values of the surface energy parameter $\lambda = \ell/g$ and with the gradient coefficient product being $c\cdot d = 0.05$ and $c\cdot d = 0.1$, respectively. The obtained results demonstrate that for small values of the product $c\cdot d$ ($c\cdot d \leqslant 0.05$), the surface energy parameter $\lambda = \ell/g$ does not affect the flexural behavior of the gradient elastic beam. For $c\cdot d > 0.05$, the deflection increases as the surface energy parameter $\lambda = \ell/g$ increases.

As a second case, the non-classical boundary conditions are assumed to be $u''(L) = u'''(0) = 0$ for the case of $\lambda = 0$ and $u''(L) = \ell u''(0) + g^2u'''(0) = 0$ for the case of $\lambda \neq 0$. Use of the above boundary conditions in conjunction with a particular solution $u_p = -(q/24EI)x^4$, enable one to determine the constants c_1–c_6 of the homogeneous part of the solution u_h given by Eq. (40). They are

$$c_1 = qL/6EI, \quad c_2 = -\frac{qL^2}{4EI}(2(c\cdot d)^2+1), \quad c_3 = \frac{qL^3(c\cdot d)^2}{EI},$$
$$c_4 = -\left(\frac{q(c\cdot d)^3L^4}{EI}\right)\mathrm{sech}\left(\frac{1}{c\cdot d}\right)\left[c\cdot d+\sinh\left(\frac{1}{c\cdot d}\right)\right], \tag{43}$$
$$c_5 = -\left(\frac{q}{EI}\right)\left[\frac{-1+c\cdot d\mathrm{e}^{1/(c\cdot d)}}{c\cdot d(1+\mathrm{e}^{2/(c\cdot d)})}\right], \quad c_6 = \left(\frac{q}{EI}\right)\left(\frac{\mathrm{e}^{1/(c\cdot d)}}{c\cdot d}\right)\left(\frac{c\cdot d+\mathrm{e}^{1/(c\cdot d)}}{1+\mathrm{e}^{2/(c\cdot d)}}\right)$$

for the case of $\lambda = 0$, and

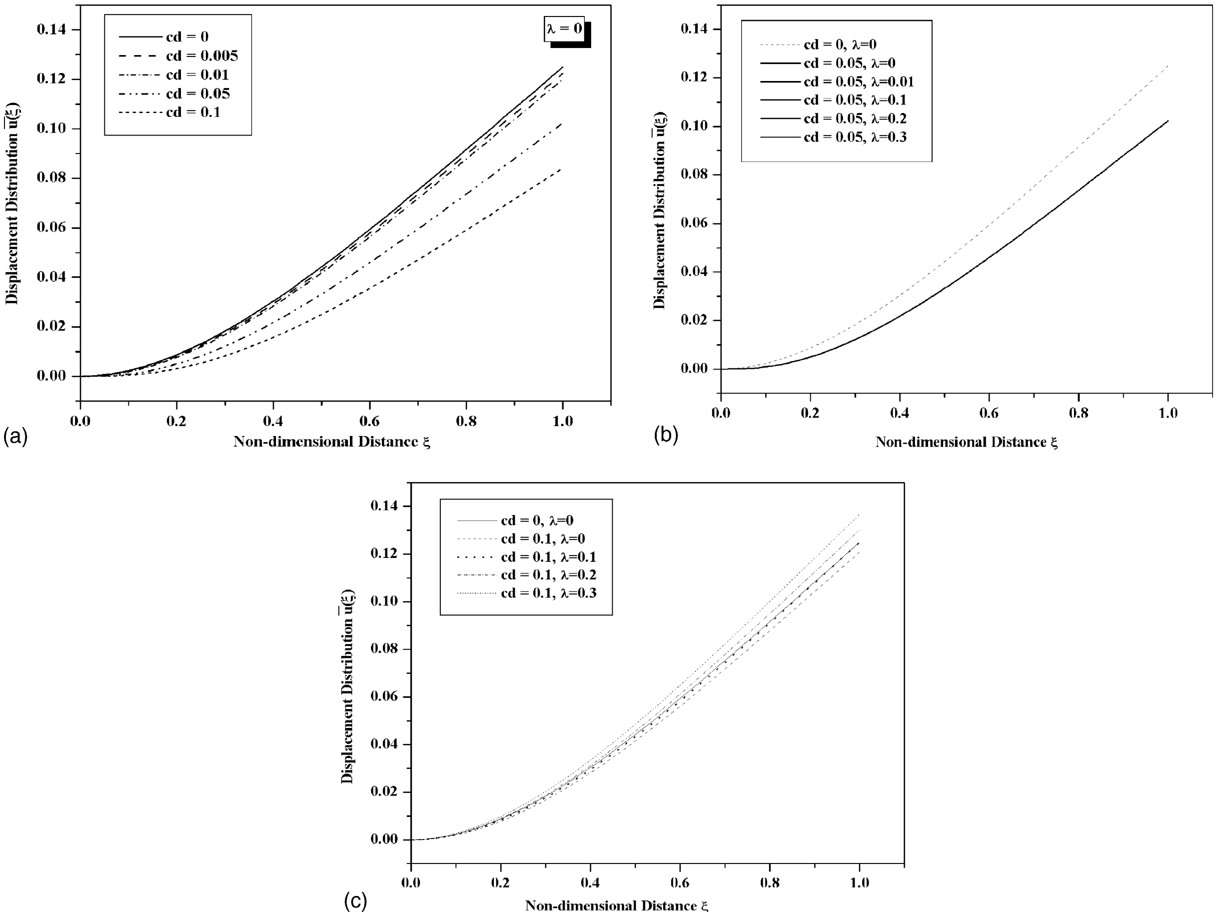

Fig. 2. (a) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of $c \cdot d$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(0) = u'''(L) = 0$. (b) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of $\lambda = \ell/g$ and $c \cdot d = 0.05$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(0) = \ell u''(L) + g^2 u'''(L) = 0$. (c) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of λ and $c \cdot d = 0.1$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(0) = \ell u''(L) + g^2 u'''(L) = 0$.

$$
\begin{aligned}
& c_1 = qL/6EI, \quad c_2 = -\frac{qL^2}{4EI}(2(c \cdot d)^2 + 1), \\
& c_3 = \left(\frac{q(c \cdot d)L^3}{EI}\right)\left[\frac{-2(c \cdot d)^2\lambda + [\lambda + 2(c \cdot d)(-1 + (c \cdot d)\lambda)]\cosh\left(\frac{1}{c \cdot d}\right)}{-2\cosh\left(\frac{1}{c \cdot d}\right) + 2\lambda\sinh\left(\frac{1}{c \cdot d}\right)}\right], \\
& c_4 = \left(\frac{q(c \cdot d)^2L^4}{EI}\right)\left[\frac{-2(c \cdot d)^2 + [\lambda + 2(c \cdot d)(-1 + (c \cdot d)\lambda)]\sinh\left(\frac{1}{c \cdot d}\right)}{2\left[\cosh\left(\frac{1}{c \cdot d}\right) - \lambda\sinh\left(\frac{1}{c \cdot d}\right)\right]}\right], \\
& c_5 = \left(\frac{q}{EI}\right)\left[\frac{-\lambda + 2(c \cdot d)\left(1 + (c \cdot d)e^{1/(c \cdot d)}(-1 + \lambda) - (c \cdot d)\lambda\right)}{2(c \cdot d)^2\left(-1 + e^{2/(c \cdot d)}(-1 + \lambda) - \lambda\right)}\right], \\
& c_6 = \left(\frac{q}{EI}\right)\left[\frac{e^{1/(c \cdot d)}\left[-2(c \cdot d)^2(1 + \lambda) + (2(c \cdot d)(-1 + (c \cdot d)\lambda) + \lambda)e^{1/(c \cdot d)}\right]}{2(c \cdot d)^2\left(-1 + e^{2/(c \cdot d)}(-1 + \lambda) - \lambda\right)}\right]
\end{aligned}
\tag{44}
$$

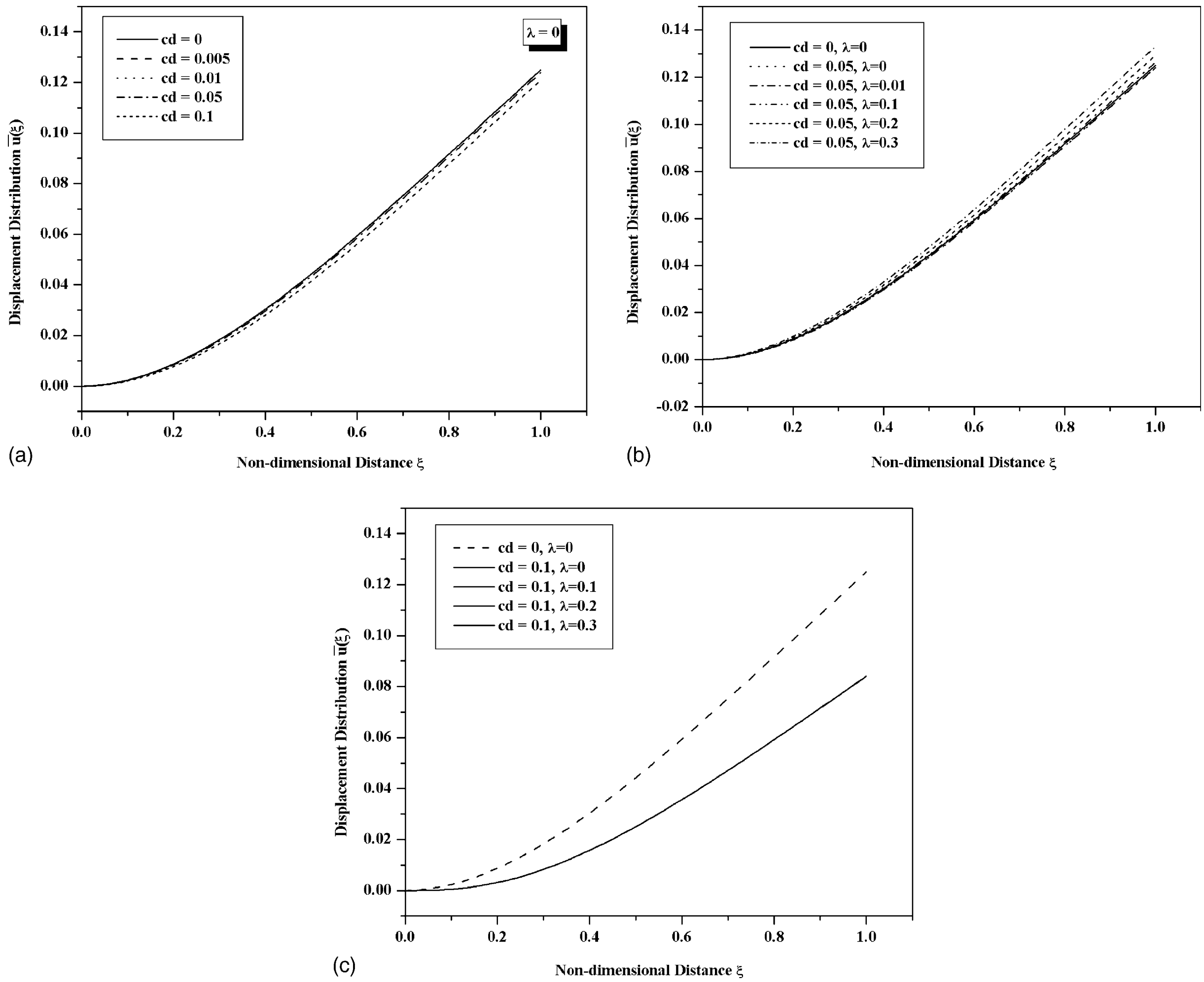

Fig. 3. (a) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of $c \cdot d$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(L) = u'''(0) = 0$. (b) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of λ and $c \cdot d = 0.05$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(L) = \ell u''(0) + g^2 u'''(0) = 0$. (c) Displacement distribution $\bar{u}(\xi)$ of a beam in bending for various values of λ and $c \cdot d = 0.1$. The classical boundary conditions are $u(0) = u'(0) = 0$ and $M(L) = V(L) = 0$ and the non-classical ones $u''(L) = \ell u''(0) + g^2 u'''(0) = 0$.

for the case of $\lambda \neq 0$.

Fig. 3(a) shows the variation of the beam deflection $\bar{u}(\xi)$ versus ξ for various values of the gradient coefficient product $c \cdot d = (g/D) \cdot (D/L)$ including the value $c \cdot d = 0$, which corresponds to the classical elastic case. Fig. 3(a) shows that the deflection of the gradient beam without surface energy decreases as the product $c \cdot d$ increases. Figs. 3(b) and (c) show the variation of the beam deflection $\bar{u}(\xi)$ versus ξ for various values of the surface energy parameter $\lambda = \ell/g$ and with gradient coefficient product being $c \cdot d = 0.05$ and $c \cdot d = 0.1$, respectively. The obtained results demonstrate that for large values of the product $c \cdot d$ ($c \cdot d > 0.05$) the surface energy parameter $\lambda = \ell/g$ does not affect the flexural behavior of the gradient elastic beam. For $c \cdot d \leqslant 0.05$ the deflection increases as the surface energy parameter $\lambda = \ell/g$ increases.

The above results on beam bending are in agreement with those of the Cosserat (micropolar elasticity) theory (Krishna Reddy and Venkatasubramanian, 1978; Lakes, 1983, 1986, 1995; Anderson and Lakes, 1994) for the case of zero surface energy. Thus non-classical flexural rigidity is always greater than the classical one for both theories. However, while in the present case increase of the beam slenderness results in a decrease of the flexural rigidity, the opposite happens in the case of Cosserat's theory (Krishna Reddy and Venkatasubramanian, 1978; Lakes, 1983, 1986, 1995; Anderson and Lakes, 1994).

7. Solution of boundary value problems for buckling

Consider a simply supported beam under the action of an axial compressive force P. The governing equation of a beam in buckling is given by

$$EI(u^{\mathrm{IV}} - g^2 u^{\mathrm{VI}}) + Pu'' = 0 \tag{45}$$

The solution of Eq. (45) is of the form

$$u = c_1 x + c_2 + c_3 \sin \xi x + c_4 \cos \xi x + c_5 \sinh \theta x + c_6 \cosh \theta x \tag{46}$$

where

$$\begin{aligned} \xi &= \left(1\Big/\sqrt{2(c\cdot d)^2 L^2}\right)\sqrt{\sqrt{1+4(c\cdot d)^2 L^2 k^2} - 1} \\ \theta &= \left(1\Big/\sqrt{2(c\cdot d)^2 L^2}\right)\sqrt{1+\sqrt{1+4(c\cdot d)^2 L^2 k^2}} \\ k^2 &= P/EI \end{aligned} \tag{47}$$

and c_1–c_6 are constants of integration to be determined from the boundary conditions of the problem.

The classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ implying, the first two that $\delta u(0) = \delta u(L) = 0$ and the second two, on account of Eq. (36), that $u''(0) - g^2 u^{\mathrm{IV}}(0) = u''(L) - g^2 u^{\mathrm{IV}}(L) = 0$. Thus, Eqs. $(36)_{1,2}$ are satisfied. The non-classical boundary conditions are assumed to be $u''(0) = u''(L) = 0$, which satisfy Eq. $(36)_3$ and further imply that $u^{\mathrm{IV}}(0) = u^{\mathrm{IV}}(L) = 0$ for the case of $\lambda = 0$.

Thus, the boundary conditions of the problem are $u(0) = u(L) = 0$, $u''(0) = u''(L) = 0$ and $u^{\mathrm{IV}}(0) = u^{\mathrm{IV}}(L) = 0$ and serve to determine the constants c_1–c_6 of Eq. (46). Indeed one easily finds that $c_1 = c_2 = c_4 = c_5 = c_6 = 0$, the buckling shape has the form

$$u(x) = c_3 \sin \xi x \tag{48}$$

i.e. the same as in the classical case and the buckling condition reads

$$\sin \xi L = 0 \tag{49}$$

Eq. (49) is satisfied for $\xi L = n\pi$ $(n = 1, 2, \ldots)$ and in view of Eq. $(47)_1$ one can obtain the first critical load for $n = 1$ in the form

$$P_{\mathrm{cr}} = (EI/4(c\cdot d)^2 L^2)[(1 + 2(c\cdot d)^2\pi^2)^2 - 1] \tag{50}$$

This expression for $c \cdot d = 0$ reduces through a limiting process to

$$P^0_{\mathrm{cr}} = \pi^2 EI/L^2 \tag{51}$$

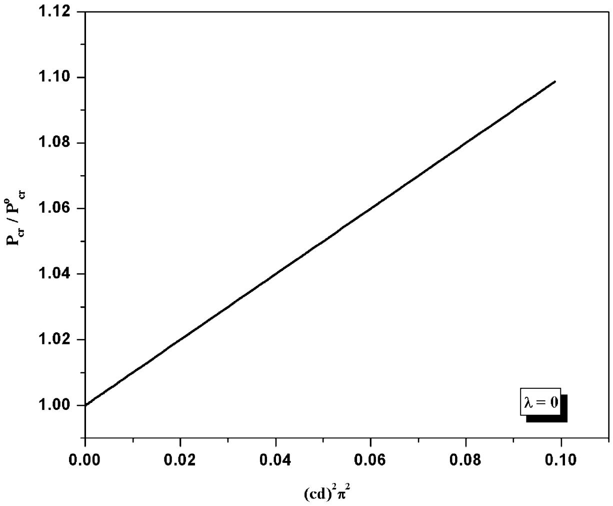

Fig. 4. Variation of the dimensionless buckling load P_{cr}/P_{cr}^0 of the gradient case as a function of $(c \cdot d)^2\pi^2$. The classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ and the non-classical ones $u''(0) = u''(L) = 0$.

which is the critical (buckling) load of the classical case. Fig. 4 depicts the variation of the ratio $P_{cr}/P_{cr}^0 = 1 + (c \cdot d)^2\pi^2$ as a function of the gradient term $(c \cdot d)^2\pi^2$, where $c \cdot d = (g/D) \cdot (D/L)$. It is apparent that the buckling load increases for increasing values of the gradient coefficient product $c \cdot d$ with the classical elastic critical load being a lower bound.

For the second case, the classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ implying, the first two that $\delta u(0) = \delta u(L) = 0$ and the second two that $u''(0) - g^2 u^{IV}(0) = u''(L) - g^2 u^{IV}(L) = 0$. Thus, Eqs. $(36)_{1,2}$ are satisfied. The non-classical boundary conditions which satisfy Eq. $(36)_3$, are assumed to be $u'''(0) = u'''(L) = 0$ for the case of $\lambda = 0$ and $\ell u''(0) + g^2 u'''(0) = \ell u''(L) + g^2 u'''(L) = 0$ for the case of $\lambda \neq 0$.

In order to have a non-trivial solution, the determinant of the coefficient matrix of the unknowns c_1–c_6 should satisfy the following condition

$$\det = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0 \tag{52}$$

where

$$\begin{aligned}
&a_{12} = a_{21} = a_{14} = a_{23} = 0 \\
&a_{11} = -\xi^3, \quad a_{13} = \theta^3, \quad a_{22} = -(c \cdot d)L^4\xi^4 - \xi^2, \quad a_{24} = -(c \cdot d)L^4\theta^4 + \theta^2 \\
&a_{31} = -\xi^3\cos(\xi L), \quad a_{32} = \xi^3\sin(\xi L), \quad a_{33} = \theta^3\cosh(\theta L), \quad a_{34} = \theta^3\sinh(\theta L) \\
&a_{41} = -(\xi^2 + (c \cdot d)^2L^4\xi^4)\sin(\xi L), \quad a_{42} = -(\xi^2 + (c \cdot d)^2L^4\xi^4)\cos(\xi L), \\
&a_{43} = (\theta^2 - (c \cdot d)^2L^4\theta^4)\sinh(\theta L), \quad a_{44} = (\theta^2 - (c \cdot d)^2L^4\theta^4)\cosh(\theta L)
\end{aligned} \tag{53}$$

for the case of $\lambda = 0$, and

$$\begin{aligned}
&a_{11} = a_{13} = 0, \quad a_{12} = -(\xi^2 + (c \cdot d)^2 L^4 \xi^4), \quad a_{14} = \theta^2 - (c \cdot d)^2 L^4 \theta^4, \\
&a_{21} = -(\xi^2 + (c \cdot d)^2 L^4 \xi^4) \sin(\xi L), \quad a_{22} = -(\xi^2 + (c \cdot d)^2 L^4 \xi^4) \cos(\xi L), \\
&a_{23} = (\theta^2 - (c \cdot d)^2 L^4 \theta^4) \sinh(\theta L), \quad a_{22} = (\theta^2 - (c \cdot d)^2 L^4 \theta^4) \cosh(\theta L), \\
&a_{31} = -(c \cdot d)^2 L^3 \xi^3, \quad a_{32} = -\lambda (c \cdot d) L \xi^2, \quad a_{33} = (c \cdot d)^2 L^3 \theta^3, \quad a_{34} = \lambda (c \cdot d) L \theta^2, \\
&a_{41} = -\lambda (c \cdot d) L \xi^2 \sin(\xi L) - (c \cdot d)^2 L^3 \xi^3 \cos(\xi L), \quad a_{42} = -\lambda (c \cdot d) L \xi^2 \cos(\xi L) + (c \cdot d)^2 L^3 \xi^3 \sin(\xi L), \\
&a_{43} = \lambda (c \cdot d) L \theta^2 \sinh(\theta L) + (c \cdot d)^2 L^3 \theta^3 \cosh(\theta L), \quad a_{44} = \lambda (c \cdot d) L \theta^2 \cosh(\theta L) + (c \cdot d)^2 L^3 \theta^3 \sinh(\theta L)
\end{aligned} \tag{54}$$

for the case of $\lambda \neq 0$.

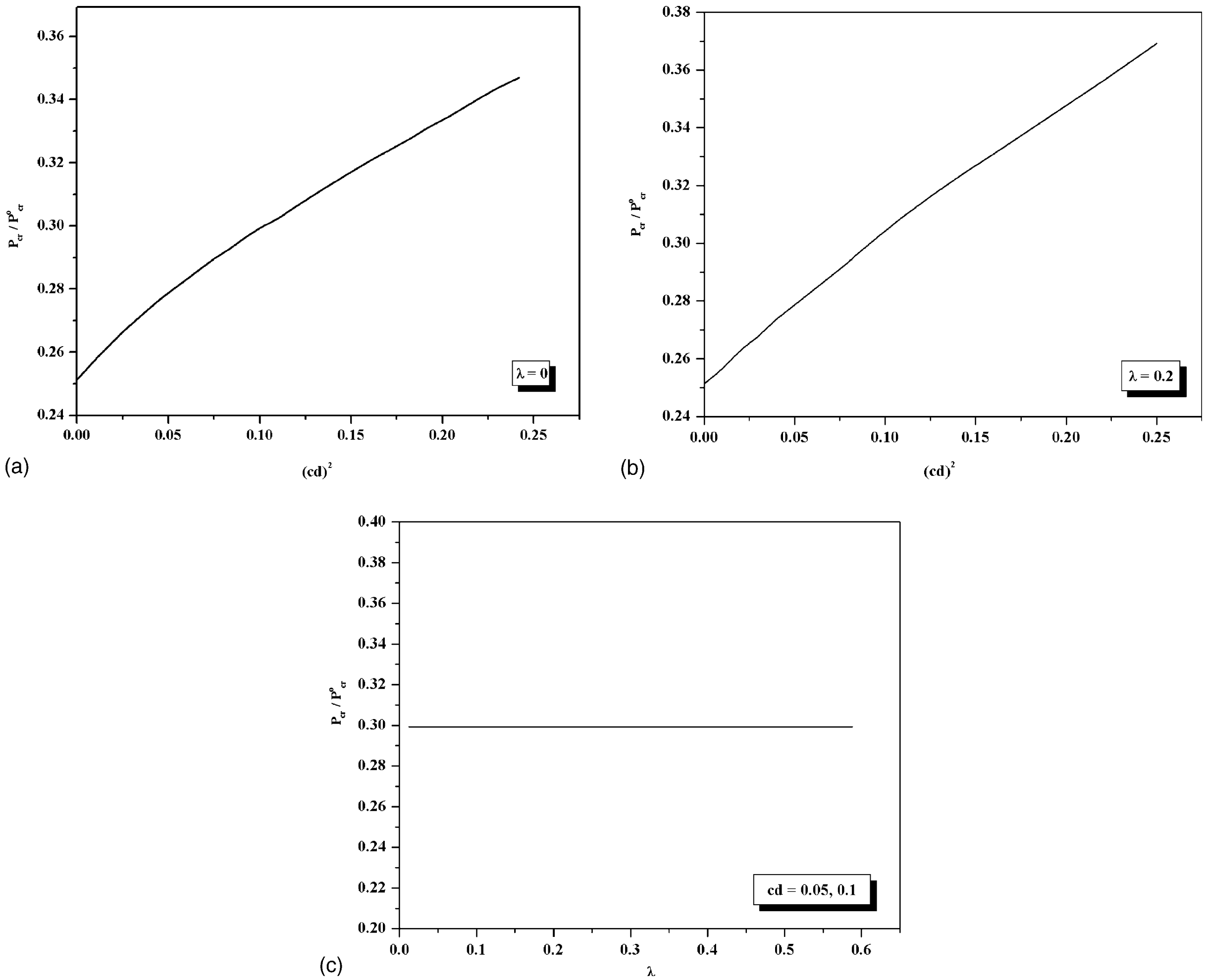

Fig. 5. (a) Variation of the dimensionless buckling load P_{cr}/P_{cr}^0 of the gradient case as a function of $(c \cdot d)^2$. The classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ and the non-classical ones $u'''(0) = u'''(L) = 0$. (b) Variation of the dimensionless buckling load P_{cr}/P_{cr}^0 of the gradient case as a function of $(c \cdot d)^2$ and $\lambda = 0.2$. The classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ and the non-classical ones $\ell u''(0) + g^2 u'''(0) = \ell u''(L) + g^2 u'''(L) = 0$. (c) Variation of the dimensionless buckling load of the gradient case as a function of λ and $c \cdot d = 0.1$. The classical boundary conditions are $u(0) = u(L) = 0$ and $M(0) = M(L) = 0$ and the non-classical ones $\ell u''(0) + g^2 u'''(0) = \ell u''(L) + g^2 u'''(L) = 0$.

Fig. 5(a) depicts the variation of the ratio P_{cr}/P_{cr}^0 as a function of the gradient coefficient product $(c \cdot d)^2$ and $\lambda = 0$. It is apparent that the buckling load increases for increasing values of the gradient coefficient product $c \cdot d$. Fig. 5(b) shows the variation of the ratio P_{cr}/P_{cr}^0 as a function of the gradient coefficient product $(c \cdot d)^2$ and the surface energy parameter $\lambda = 0.2$. Fig. 5(c) shows the variation of the ratio P_{cr}/P_{cr}^0 as a function of the surface energy parameter λ and the gradient coefficient product $c \cdot d = 0.1$. It is observed that the surface energy effect is negligible and that for some other non-classical boundary conditions the normalized critical load versus normalized gradient coefficient relation may be slightly non-linear.

8. Conclusions

On the basis of the preceding discussion, the following conclusions can be stated:

(i) Using a simple theory of gradient elasticity with surface energy, the governing equations of beam bending and buckling and the corresponding boundary conditions (classical and non-classical) have been derived.

(ii) The boundary conditions have been derived from a variational statement constructed either directly or indirectly with the aid of the method of weighted residuals. The latter approach does not require any knowledge of the strain energy, which is obtained here as a byproduct.

(iii) A characteristic boundary value problem of beam in bending has been solved and its gradient elastic solution for the beam deflection has been found to decrease (but not very significantly) for increasing values of the gradient coefficient $c \cdot d$ with the classical elastic solution being an upper bound. On the other hand the surface energy effect has been found to be rather small, dependent on the non-classical boundary conditions and to lead to either increasing or decreasing displacements depending on the value of the gradient coefficient.

(iv) A characteristic boundary value problem of beam buckling has been solved and its gradient elastic solution for the critical (buckling) load has been found to increase for increasing values of the gradient coefficient with the classical elastic critical load being a lower bound. The surface energy effect has been found to be insignificant.

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References

Altan, B.S., Aifantis, E.C., 1992. On the structure of mode-III crack tip in gradient elasticity. Scripta Metall. Mater. 26, 319–324.

Altan, B.S., Evensen, H., Aifantis, E.C., 1996. Longitudinal vibrations of a beam: a gradient elasticity approach. Mech. Res. Commun. 23, 35–40.

Anderson, W.B., Lakes, R.S., 1994. Size effects due to Cosserat elasticity and surface damage in closed-cell polymethacrylimide foam. J. Mater. Sci. 29, 6413–6419.

Casal, P., 1972. La theorie du second gradient et la capillarite'. C.R. Acad Sci. A (274), 1571–1574.

Chang, C.S., Gao, J., 1997. Wave propagation in granular rod using high-gradient theory. J. Eng. Mech., ASCE 123, 52–59.

Cosserat, E., Cosserat, F., 1909. Theorie de Corps Deformables. Editions A. Hermann, Paris.

Eringen, A.C., 1966. Linear theory of micropolar elasticity. J. Math. Mech. 15, 909–923.

Exadaktylos, G.E., Vardoulakis, I., 2001. Microstructure in linear elasticity and scale effects: a reconsideration of basic rock mechanics and rock fracture mechanics. Tectonophysics 335, 81–109.

Exadaktylos, G., Vardoulakis, I., Aifantis, E.C., 1996. Cracks in gradient elastic bodies with surface energy. Int. J. Fract. 79, 107–119.

Georgiadis, H.G., Vardoulakis, I., 1998. Anti-plane shear Lamb's problem treated by gradient elasticity with surface energy. Wave Motion 28, 353–366.

Krishna Reddy, G.V., Venkatasubramanian, N.K., 1978. On the flexural rigidity of a micropolar elastic circular cylinder. J. Appl. Mech., ASME 45, 429–431.

Lakes, R.S., 1983. Size effects and micromechanics of a porous solid. J. Mater. Sci. 18, 2572–2580.

Lakes, R.S., 1986. Experimental microelasticity of two porous solids. Int. J. Solids Struct. 22, 55–63.

Lakes, R., 1995. Experimental methods for study of Cosserat elastic solids and other generalized elastic continua. In: Muhlhaus, H.B. (Ed.), Continuum Models for Materials with Microstructure. John Wiley & Sons, Chichester, pp. 1–25.

Lanczos, C., 1970. The Variational Principles of Mechanics. University of Toronto Press, Toronto.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch, Rat. Mech. Anal. 16, 51–78.

Ru, C.Q., Aifantis, E.C., 1993. A simple approach to solve boundary value problems in gradient elasticity. Acta Mech. 101, 59–68.

Tiersten, H.F., Bleustein, J.L., 1974. Generalized elastic continua. In: Herrmann, G. (Ed.), R.D. Mindlin and Applied Mechanics. Pergamon Press, New York, pp. 67–103.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity, third ed. McGraw-Hill Book Co., Inc., New York.

Tsagrakis, I.A., 2001. The role of gradients in elasticity and plasticity: analytical and numerical applications. Doctoral Dissertation, Aristotle University of Thessaloniki, Thessaloniki, Greece (in Greek).

Tsepoura, K.G., Papargyri-Beskou, S., Polyzos, D., Beskos, D.E., 2002. Static and dynamic analysis of a gradient elastic bars in tension. Arch. Appl. Mech. 72, 483–497.

Vardoulakis, I., Exadaktylos, G., Kourkoulis, S.K., 1998. Bending of marble with intrinsic length scales: a gradient theory with surface energy and size effects. J. de Physique IV 8, 399–406.

Vardoulakis, I., Sulem, J., 1995. Bifurcation Analysis in Geomechanics. Blackie/Chapman and Hall, London.